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Wronski Algebra Systems and Residues

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Abstract. We apply the theory of residues to characterize the substitutes for the sheaves of principal parts on Gorenstein, projective curves introduced by Laksov and Thorup [**6**], and we compare these substitutes with those introduced by the author [**2, 3**]. Our characterization extends a characterization by Atiyah of the sheaves of first order principal parts with coefficients in an invertible sheaf on smooth curves [**1**, **Prop. 12**].

1. Introduction

The sheaves of principal parts play an important role in the study of the projective geometry of smooth curves in any characteristic. One of the most important properties of these sheaves is that they are locally free over a smooth curve. However, they fail to be locally free, or even torsion-free, over a singular curve.

Recently, locally free substitutes for the sheaves of principal parts over singular curves were independently introduced by Laksov and Thorup [**6**] and the author [**2, 3**]. It is one of the goals of this article to compare them. We will see that the two substitutes seldom coincide for singular curves (Corollary 3.5.) The difference does not come as a surprise: even though the sheaves introduced in [**6**] are quite natural, their introduction makes use of the normalization map of the curve, via Rosenlicht's local characterization of the dualizing sheaf [**9**, p. 76]. Thus we might have expected that their definition does not extend to families, in contrast with the sheaves defined in [**2, 3**].

Our main technical result is Theorem 3.2, where we give a charac-

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terization via residues of Laksov's and Thorup's substitutes when these are taken with coefficients in an invertible sheaf. Of course such characterization applies to a smooth curve, and thus it must extend Atiyah's characterization [**1**, Prop. 12] of the sheaf of first order principal parts with coefficients in an invertible sheaf to higher orders. Atiyah worked mainly with Chern classes, but the trace map provides a way to compare his characterization with ours (Remark 3.4.)

The remaining of this article is divided into two sections as follows: in Section 2 we introduce the notion of a Wronski algebra system, and describe the systems constructed by Laksov and Thorup and by the author; in Section 3 we prove our main technical result, and use it to compare Laksov's and Thorup's substitutes for the sheaves of principal parts with the author's.

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2. Wronski algebra systems

Let S be a Noetherian scheme. Let $f\colon X \to S$ be a flat morphism of schemes whose geometric fibres are reduced, Gorenstein schemes of pure dimension 1. We say that f, or X/S, is a *family of curves.* For each $n \geq 0$, let P^n be the sheaf of relative n-th order principal parts on X over S. Recall that there are canonical surjective algebra homomorphisms $p^n\colon P^n \to P^{n-1}$ for every $n > 0$. Let Ω^n denote the kernel of p^n for every $n > 0$. Of course, Ω^1 is the sheaf of relative Kähler differentials on X over S. Recall that there are two canonical $\mathcal{O}_X$-algebra structures on P^n for each $n \geq 0$. By convention, we call them the *left* and *right structures* of P^n, and denote the structure homomorphisms by

$$\delta_l\colon \mathcal{O}_X \to P^n \quad \text{and} \quad \delta_r\colon \mathcal{O}_X \to P^n,$$

respectively. We don't emphasize the dependence on n in the above notation, since the p^n are natural $\mathcal{O}_X$-algebra homomorphisms with respect to both left and right structures. Recall that both $\mathcal{O}_X$-module structures coincide on Ω^n for every $n > 0$. Put $\delta := \delta_r - \delta_l$.

Let ω be an invertible sheaf on X. Assume that there is a homomorphism $\eta^1 \colon \Omega^1 \to \omega$ that is bijective on the S-smooth locus of X. Then η^1 induces homomorphisms $\eta^n \colon \Omega^n \to \omega^{\otimes n}$ for every $n > 0$ that are also bijective on the S-smooth locus of X (the argument used in the proof of [**3**, Prop. 2.3] applies also here.)

A collection of sheaves of algebras $F := (F^n, n \geq 0)$ on X, together with algebra homomorphisms

$$\psi^n \colon P^n \to F^n,$$
$$q^n \colon F^n \to F^{n-1},$$

and an $\mathcal{O}_X$-bimodule homomorphism

$$\alpha^n \colon \omega^{\otimes n} \to F^n$$

for every $n \geq 0$, is said to be a *Wronski algebra system on X over S* if the following conditions are satisfied:

(1) ψ^0 is an isomorphism;

(2) the diagram of maps

$$\begin{tikzcd} 0 \arrow[r] & \Omega^n \arrow[r] \arrow[d, "\eta^n"'] & P^n \arrow[r, "p^n"] \arrow[d, "\psi^n"'] & P^{n-1} \arrow[r] \arrow[d, "\psi^{n-1}"'] & 0 \\ 0 \arrow[r] & \omega^{\otimes n} \arrow[r, "\alpha^n"] & F^n \arrow[r, "q^n"] & F^{n-1} \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows for every $n > 0$.

Note that ψ^n induces left and right $\mathcal{O}_X$-algebra structures on F^n for each $n \geq 0$. By definition, α^n is $\mathcal{O}_X$-linear with respect to both left and right structures on F^n. Moreover, note that the ψ^n are isomorphisms on the S-smooth locus of X, since the η^n have this property and ψ^0 is an isomorphism. Finally, note that F^n is locally free of rank $n+1$ under both left and right $\mathcal{O}_X$-algebra structures for every $n \geq 0$, since $F^0 = \mathcal{O}_X$ and ω is invertible.

Assume for this paragraph that the geometric fibres of X/S are local complete intersections. Let ω denote the canonical sheaf on X/S (see [**3**, Section 2] or [**7**].) There is a canonical homomorphism $\eta^1\colon \Omega^1 \to \omega$ that is bijective on the S-smooth locus of X [**3**, Section 2]. By [**3**, Theorem 2.6], there is a Wronski algebra system $(F^n, n \geq 0)$ on X over S. More precisely, there is a unique Wronski algebra system, denoted in this article by $Q := (Q^n, n \geq 0)$, if we choose it with the additional property that its formation commutes with base change on the class of all families X/S of local complete intersection curves.

Assume for the rest of Section 2 that S is the spectrum of an algebraically closed field k, and X is irreducible. Let $\pi\colon \tilde{X} \to X$ denote the normalization of X. Let K denote the field of meromorphic functions of X and $\tilde{X}$. For any sheaf $\mathcal{F}$ on X we denote by $\mathcal{F}_K$ its stalk at the generic point of X. If V is any K-vector space, we also denote by V the corresponding constant sheaf on X. For each $q \in \tilde{X}$ denote by res_q the Tate residue map at q (see [**4**, p. 247].) Let ω denote the $\mathcal{O}_X$-submodule of regular differentials of Ω^1_K (see [**9**, p. 76].) If $p \in X$, then

$$\omega_p = \{\nu \in \Omega^1_K;\ \sum_{q\in\pi^{-1}(p)} res_q(f\nu) = 0 \text{ for all } f \in \mathcal{O}_{X,p}\}. \tag{2.0.1}$$

Of course, there is a canonical homomorphism $\eta^1\colon \Omega^1 \to \omega$, and η^1 is bijective on the smooth locus of X.

In [**6**, Section 6], Laksov and Thorup defined a Wronski algebra system, denoted in this article by $R := (R^n, n \geq 0)$, on X. We describe locally the sheaves R^n below. Let $p \in X$. There are an open neighbourhood U of p in X and a rational function $t \in K$ such that t is regular on $\pi^{-1}(U)$ and the differential dt is a basis for the sheaf of Kähler differentials of $\pi^{-1}(U)$. Choose U small enough that ω is free on U. Then ω_U is generated by the meromorphic differential dt/h, for some regular function h on U, and the restriction of R^n to U is the $\mathcal{O}_U$-subalgebra of P^n_K generated as a free left $\mathcal{O}_U$-submodule of P^n_K by the basis

$$1, (\delta t/h), \ldots, (\delta t/h)^n$$

for every $n \geq 0$.

3. The characterization

Let S be a Noetherian scheme. Let $f\colon X \to S$ be a projective family of curves. Let ω be a relative dualizing sheaf on X over S, and let $T\colon R^1 f_* \omega \to \mathcal{O}_S$ be a trace map. The sheaf ω is invertible, since the fibres of X/S are Gorenstein. Assume that there is a homomorphism $\eta^1\colon \Omega^1 \to \omega$ that is bijective on the S-smooth locus of X. Finally, assume that there is a Wronski algebra system $F := (F^n, n \geq 0)$ on X over S.

Let L be an invertible sheaf on X. For each $n \geq 0$, let $F^n(L)$ denote the tensor product of F^n and L with respect to the right $\mathcal{O}_X$-structure of F^n. For every $n > 0$ we have an exact sequence,

$$0 \to \omega^{\otimes n} \otimes L \xrightarrow{\alpha^n \otimes \mathrm{id}_L} F^n(L) \xrightarrow{q^n \otimes \mathrm{id}_L} F^{n-1}(L) \to 0,$$

that we regard as a sequence of left $\mathcal{O}_X$-modules. The above exact sequence gives rise to a global section $[F^n(L)]$ of

$$Ext^1_f(F^{n-1}(L), \omega^{\otimes n} \otimes L).$$

Since $F^{n-1}(L)$ is locally free, then

$$Ext^1_f(F^{n-1}(L), \omega^{\otimes n} \otimes L) = R^1 f_* \underline{Hom}(F^{n-1}(L), \omega^{\otimes n} \otimes L).$$

From the inclusion

$$\omega^{\otimes n-1} \otimes L \xrightarrow{\alpha^{n-1} \otimes \mathrm{id}_L} F^{n-1}(L),$$

we obtain a homomorphism

$$R^1 f_* \underline{Hom}(F^{n-1}(L), \omega^{\otimes n} \otimes L) \longrightarrow R^1 f_* \underline{Hom}(\omega^{\otimes n-1} \otimes L, \omega^{\otimes n} \otimes L). \tag{3.0.1}$$

We also have the canonical identification:

$$R^1 f_* \underline{Hom}(\omega^{\otimes n-1} \otimes L, \omega^{\otimes n} \otimes L) = R^1 f_* \omega.$$

By composing (3.0.1) with the trace map T, we finally obtain a regular function $\sigma^n_F(L)$ on S corresponding to $[F^n(L)]$.

The function $\sigma^n_F(L)$ depends on the choice of ω and the trace map T, but its class in $H^0(S,\mathcal{O}_S)/H^0(S,\mathcal{O}_S)^*$ is independent. Moreover, this class depends only on the equivalence class of F. Note that in most cases the function $\sigma^n_F(L)$ determines the extension $[F^n(L)]$, as the next proposition shows.

Proposition 3.1. *If $R^1f_*(\omega^{\otimes m}) = 0$ for all $2 \leq m \leq n$, then $\sigma^n_F(L)$ determines $[F^n(L)]$. In particular, if the arithmetic genus of every geometric fibre of X over S is at least 2, then the functions $(\sigma^n_F(L), n \geq 0)$ determine the Wronski system $(F^n(L), n \geq 0)$.*

Proof. Since the trace map T is an isomorphism, then $\sigma^n_F(L)$ determines $[F^n(L)]$ if and only if (3.0.1) is an isomorphism. We will show by induction on j that the inclusion $\alpha^j \otimes \mathrm{id}_L$ induces an isomorphism

$$R^1f_*\underline{Hom}(F^j(L), \omega^{\otimes n} \otimes L) \xrightarrow{\beta^j} R^1f_*\underline{Hom}(\omega^{\otimes j} \otimes L, \omega^{\otimes n} \otimes L)$$

for $0 \leq j \leq n-1$. Since the homomorphism (3.0.1) is β^{n-1}, then the proposition will follow. Since α^0 is an isomorphism, then so is β^0. Assume that β^{m-2} is an isomorphism for a certain integer m with $2 \leq m \leq n$. Note that β^{m-1} is surjective because X/S is a family of curves. On the other hand, the kernel of β^{m-1} is a quotient of

$$R^1f_*\underline{Hom}(F^{m-2}(L), \omega^{\otimes n} \otimes L),$$

and the latter sheaf is zero by the hypothesis of the proposition and the induction hypothesis. So β^{m-1} is an isomorphism, finishing the induction argument and the proof. □

Theorem 3.2. *Let X be a projective, integral curve of geometric genus g defined over an algebraically closed field k. Let ω be the dualizing sheaf on X, and let $T\colon H^1(X,\omega) \to k$ be the canonical trace map, given in terms of residues. Let L be an invertible sheaf on X of degree d. Let $R := (R^n, n \geq 0)$ be the Wronski algebra system constructed by Laksov and Thorup. Then*

$$\sigma^n_R(L) = ((n-1)(1-g) - d)\mathbf{1}_k$$

for every $n > 0$, where $\mathbf{1}_k$ denotes the unit of k.

Proof. (For details on the theory of residues on singular varieties, and in particular for the construction of the trace map from residues, see [**8**].) Tensoring the canonical flasque resolution of $\mathcal{O}_X$ by meromorphic functions,

$$0 \to \mathcal{O}_X \to K \to \bigoplus_{p\in X} j_{p*}\frac{K}{\mathcal{O}_{X,p}} \to 0, \tag{3.2.1}$$

with $\omega^{\otimes n}\otimes L$, we obtain a flasque resolution of $\omega^{\otimes n}\otimes L$. (The morphism j_p is the inclusion of p in X.) By means of this flasque resolution, we have:

$$Ext^1(R^{n-1}(L), \omega^{\otimes n}\otimes L) = \frac{\bigoplus Hom(R^{n-1}(L)_p, \frac{(\Omega^1_K)^{\otimes n}\otimes L_K}{\omega_p^{\otimes n}\otimes L_p})}{Hom(R^{n-1}(L)_K, (\Omega^1_K)^{\otimes n}\otimes L_K)},$$

where the above direct sum runs over all $p \in X$. To obtain an element in

$$\bigoplus_{p\in X} Hom(R^{n-1}(L)_p, \frac{(\Omega^1_K)^{\otimes n}\otimes L_K}{\omega_p^{\otimes n}\otimes L_p})$$

representing $[R^n(L)]$ in $Ext^1(R^{n-1}(L), \omega^{\otimes n}\otimes L)$ we proceed as follows. Let t be a separating variable for K over k. Let dt denote the associated differential in Ω^1_K. Then dt is a generator of Ω^1_K over K. Let ψ_K be a K-generator of L_K. Then

$$\psi_K, \delta t\otimes\psi_K, (\delta t)^2\otimes\psi_K, \ldots, (\delta t)^n\otimes\psi_K \tag{3.2.2}$$

form a left K-basis of $R^n(L)_K$, where $\otimes$ denotes the tensor product with respect to the right $\mathcal{O}_X$-algebra structure of R^n. A splitting ρ of the inclusion of left K-vector spaces

$$(\Omega^1)_K^{\otimes n}\otimes L_K \longrightarrow R^n(L)_K$$

is then defined by mapping the basis elements

$$\psi_K, \delta t\otimes\psi_K, \ldots, (\delta t)^{n-1}\otimes\psi_K$$

to 0 and $(\delta t)^n\otimes\psi_K$ to $(dt)^{\otimes n}\otimes\psi_K$. Consider now the following composition,

$$R^n(L) \to R^n(L)_K \xrightarrow{\ \rho\ } (\Omega^1_K)^{\otimes n}\otimes L_K \to \bigoplus_{p\in X} j_{p*}\frac{(\Omega^1_K)^{\otimes n}\otimes L_K}{\omega_p^{\otimes n}\otimes L_p},$$

where the first and last maps are canonical. It is clear that the above composition factors through $R^{n-1}(L)$, yielding the representative of $[R^n(L)]$ we were looking for. By composing with the inclusion

$$\omega^{\otimes n-1} \otimes L \to R^{n-1}(L)$$

we get a homomorphism

$$\omega^{\otimes n-1} \otimes L \to \bigoplus_{p\in X} j_{p*}\frac{(\Omega_K^1)^{\otimes n} \otimes L_K}{\omega_p^{\otimes n} \otimes L_p}, \tag{3.2.3}$$

that we describe locally below.

Let $p \in X$, and s_p be a local parameter at all points of $\pi^{-1}(p)$. Then ω_p is generated by ds_p/h_p for some $h_p \in \mathcal{O}_{X,p}$. Let ψ_p be a generator of L_p. Then $R^n(L)_p$ is generated as a left $\mathcal{O}_{X,p}$-module by

$$\psi_p, \frac{\delta s_p}{h_p} \otimes \psi_p, \ldots, \frac{(\delta s_p)^n}{h_p^n} \otimes \psi_p$$

Let $f_p \in K$ be such that $\psi_p = f_p \psi_K$ in L_K. We need only describe the image of

$$\frac{ds_p^{\otimes n-1}}{h_p^{n-1}} \otimes \psi_p$$

in

$$\frac{(\Omega_K^1)^{\otimes n} \otimes L_K}{\omega_p^{\otimes n} \otimes L_p}$$

to describe the localization of (3.2.3) at p. But $ds_p^{\otimes n-1}/h_p^{n-1} \otimes \psi_p$ is mapped to the quotient class whose representative is $dt^{\otimes n} \otimes \psi_K$ times the coefficient of $(\delta t)^n \otimes \psi_K$ in the expression of $(\delta s_p)^{n-1}/h_p^{n-1} \otimes \psi_p$ in $R^n(L)_K$ with respect to the basis (3.2.2). This class is easily found to be

$$\left(\frac{D_t^1(s_p)^{n-1} D_t^1(f_p) + (n-1) D_t^1(s_p)^{n-2} D_t^2(s_p) f_p}{h_p^{n-1}}\right) dt^{\otimes n} \otimes \psi_K,$$

where D_z^j denotes the Hasse derivation of order j on K with respect to a separating variable z. Tensoring (3.2.3) with $\omega^{\otimes 1-n} \otimes L^{-1}$, we get a canonical element of

$$\bigoplus_{p\in X} \frac{\Omega_K^1}{\omega_p},$$

whose component at p has

$$\left(\frac{D_t^1(f_p)}{D_t^1(s_p)f_p} + (n-1)\frac{D_t^2(s_p)}{D_t^1(s_p)^2}\right)ds_p$$

as a representative.

Let

$$res_p\colon \Omega_K^1 \to k$$

denote the residue map at p. The map res_p may be defined as the sum of the Tate residue maps res_q at all $q \in \pi^{-1}(p)$. It follows from (2.0.1) that res_p factors through Ω_K^1/ω_p. It follows from tensoring the flasque resolution (3.2.1) by ω that

$$H^1(X,\omega) = \frac{\bigoplus \frac{\Omega_K^1}{\omega_p}}{\Omega_K^1},$$

where the above direct sum runs over all $p \in X$. It follows from the residue theorem that the sum of the residue maps at all $p \in X$ induces a map $H^1(X,\omega) \to k$. The latter is the trace map T in the statement of the theorem. Thus

$$\sigma_R^n(L) = \sum_{p\in X} res_p\left(\left(\frac{D_{s_p}^1(f_p)}{f_p} + (n-1)\frac{D_t^2(s_p)}{D_t^1(s_p)^2}\right)ds_p\right).$$

Note that the residues of both

$$\frac{D_s^1(f_p)}{f_p}ds \quad\text{and}\quad \frac{D_t^2(s)}{D_t^1(s)^2}ds$$

at $q \in \pi^{-1}(p)$ do not depend on the choice of a local parameter s at q. Because

$$res_p = \sum_{q\in\pi^{-1}(p)} res_q,$$

we may assume that X is smooth, or in other words, that $X = \tilde{X}$. The proof of the theorem is thus reduced to showing the following lemma.□

Lemma 3.3. *Let X be a projective, smooth, connected curve of genus g over an algebraically closed field k. Let L be an invertible sheaf of degree d on X. Let t (resp. ψ) be a separating variable for the function field K of X over k (resp. a generator of L_K.) For every $q \in X$, let s_q (resp.*

$f_q\psi$) be a local parameter of X at q (resp. a local generator of L at q, where $f_q \in K$.) Then

$$(1)\quad \sum_{q\in X} res_q\Big(\frac{D^1_{s_q}(f_q)}{f_q}ds_q\Big) = -d\mathbf{1}_k;$$

$$(2)\quad \sum_{q\in X} res_q\Big(\frac{D^2_t(s_q)}{D^1_t(s_q)^2}ds_q\Big) = (1-g)\mathbf{1}_k.$$

Proof. For every $q \in X$, let v_q denote the valuation at q. Then

$$res_q\Big(\frac{D^1_{s_q}(f_q)}{f_q}ds_q\Big) = v_q(f_q)\mathbf{1}_k. \tag{3.3.1}$$

The first statement of the lemma is an immediate consequence of (3.3.1).

As for the second statement, recall that dt is a generator for Ω^1_K and ds_q is a generator for Ω^1_q for every $q \in X$. Of course, $ds_q = D^1_t(s_q)dt$ for every $q \in X$. Note also that

$$D^1_{s_q}(D^1_t(s_q)) = 2\frac{D^2_t(s_q)}{D^1_t(s_q)}.$$

Hence, if char.$k \neq 2$, then (3.3.1) with f_q replaced by $D^1_t(s_q)$ proves the second statement of the claim as well.

Assume now that char.$k = 2$. Let $q \in X$. First note that, by the chain rule for Hasse derivations,

$$D^1_z(s_q) = \frac{1}{D^1_{s_q}(z)} \quad\text{and}\quad \frac{D^2_z(s_q)}{D^1_z(s_q)^2} = -\frac{D^2_{s_q}(z)}{D^1_{s_q}(z)}.$$

for any separating variable z. Write $t = t_1 + t_2$, where t_1 (resp. t_2) can be expressed as a Laurent series on s_q with odd (resp. even) powers. Note that t_1 is also a separating variable. We have that

$$\frac{D^2_{s_q}(t)}{D^1_{s_q}(t)} = \frac{D^2_{s_q}(t_1)}{D^1_{s_q}(t_1)} + \frac{D^2_{s_q}(t_2)}{D^1_{s_q}(t_1)}.$$

It is easy to show that

$$res_q\Big(\frac{D^2_{s_q}(t_1)}{D^1_{s_q}(t_1)}ds_q\Big) = \frac{v_q(D^1_{s_q}(t_1))}{2}\mathbf{1}_k = \frac{v_q(D^1_{s_q}(t))}{2}\mathbf{1}_k = -\frac{v_q(D^1_t(s_q))}{2}\mathbf{1}_k.$$

On the other hand,

$$res_q(\frac{D^2_{sq}(t_2)}{D^1_{sq}(t_1)}ds_q) = 0,$$

since both $D^1_{sq}(t_1)$ and $D^2_{sq}(t_2)$ can be expressed as Laurent series on s_q involving only even powers. Hence,

$$res_q(\frac{D^2_t(s_q)^2}{D^1_t(s_q)^2}ds_q) = -res_q(\frac{D^2_{sq}(t)}{D^1_{sq}(t)}ds_q) = \frac{v_q(D^1_t(s_q))}{2}\mathbf{1}_k$$

for every $q \in X$. The second statement of the lemma is an immediate consequence of the above equality.□

Remark 3.4. Atiyah [**1**, Prop. 12] had shown that $c_1(L) = -[P^1(L)]$ when X is a smooth curve, where $c_1(L)$ is the Chern class of L, and $[P^1(L)] \in H^1(X, \Omega^1)$ represents the extension

$$0 \longrightarrow \Omega^1 \otimes L \longrightarrow P^1(L) \xrightarrow{p^n \otimes L} L \longrightarrow 0.$$

It folows from the first statement of Lemma 3.3 that $T(c_1(L)) = d\mathbf{1}_k$. So we see that Atiyah's characterization is equivalent to ours (Theorem 3.2) under the trace map. Atiyah's characterization was used by Kaji to show that a tangentially degenerate smooth curve in a projective space must be contained in some 2-plane if the characteristic of the ground field is 0 [**5**, Thm. 3.1, p. 436].

Corollary 3.5. *Let X be a projective, integral, local complete intersection curve over an algebraically closed field k. For every $p \in X$, let δ_p denote the singularity degree of p. Let $Q := (Q^n, n \geq 0)$ and $R := (R^n, n \geq 0)$ be the two Wronski algebra systems on X described in Section 2. If $\delta_p \mathbf{1}_k \neq 0$ for some $p \in X$, then the sheaves Q^n and R^n are not isomorphic as P^n-algebras for every $n \geq 2$.*

Proof. Of course it is enough to show that Q^n and R^n are not isomorphic as P^n-algebras locally around a singular point of X. Taking a partial normalization of X if necessary, we may assume that $(p_a - g)\mathbf{1}_k \neq 0$, where p_a (resp. g) is the arithmetic genus (resp. geometric genus) of X. We now observe that, given a Wronski algebra system $F := (F^m, m \geq 0)$ on X, with associated homomorphisms $\psi^m \colon P^m \to F^m$ for $m \geq 0$, then the sheaf F^m and the homomorphism ψ^m are determined by F^{m+1} and ψ^{m+1} for every $m \geq 0$. In fact, we have that F^m is the torsion-free quotient of $F^{m+1}/\psi^{m+1}(\Omega^{m+1})$, while ψ^m is the composition of the homomorphism $P^m \to F^{m+1}/\psi^{m+1}(\Omega^{m+1})$ induced by ψ^{m+1} with the quotient map onto F^m. It follows from this observation that if the sheaves Q^n and R^n are isomorphic as P^n-algebras, then so are Q^m and R^m as P^m-algebras for every $m < n$. So we are reduced to showing the corollary for $n = 2$. Moreover, it follows from the same observation that if Q^2 and R^2 are isomorphic as P^2-algebras, then the exact sequences

$$0 \longrightarrow \omega^{\otimes 2} \longrightarrow Q^2 \longrightarrow Q^1 \longrightarrow 0$$

$$0 \longrightarrow \omega^{\otimes 2} \longrightarrow R^2 \longrightarrow R^1 \longrightarrow 0$$

are isomorphic, where ω is the dualizing sheaf on X. Thus we need only show that there is an invertible sheaf L on X such that the induced exact sequence

$$0 \longrightarrow \omega^{\otimes 2} \longrightarrow Q^2(L) \longrightarrow Q^1(L) \longrightarrow 0 \tag{3.5.1}$$

is split, while

$$0 \longrightarrow \omega^{\otimes 2} \longrightarrow R^2(L) \longrightarrow R^1(L) \longrightarrow 0 \tag{3.5.2}$$

is not.

Since X is a projective, local complete intersection curve, then there is a projective family $\mathcal{X}/S$ of local complete intersection curves where:
(1) S is a connected smooth curve over k;
(2) $X = \mathcal{X}(s)$ for a certain $s \in S$;
(3) $\mathcal{X}$ is smooth over $S \setminus s$.

Since S is connected, then the geometric genus of $\mathcal{X}(t)$ is p_a for every $t \in S \setminus s$. Let $\omega_{\mathcal{X}/S}$ be the canonical sheaf on X over S. Let $\mathcal{Q} := (\mathcal{Q}^m, m \geq 0)$ be the unique Wronski algebra system on $\mathcal{X}$ over S [**3**, Prop. 4.3]. As mentioned in Section 2, we have that $\mathcal{Q}(s) = Q$. Since $\mathcal{X}$ is smooth over $S \setminus s$, then $\mathcal{Q}^n(t)$ is the sheaf of n-th order principal parts on $X(t)$ for every $t \in S \setminus s$ and every $n \geq 0$.

Replacing S by a connected, étale neighbourhood of s if necessary, we may assume that there is an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ of relative degree

$(1-p_a)$ over S. Let $L := \mathcal{L}(s)$. It follows from Theorem 3.2 that the exact sequence

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \otimes \mathcal{L}(t) \longrightarrow \mathcal{Q}^2(t)(\mathcal{L}(t)) \longrightarrow \mathcal{Q}^1(t)(\mathcal{L}(t)) \longrightarrow 0$$

is split if $t \in S \setminus s$. By semicontinuity, since (3.5.1) is the limit of the above exact sequence when t tends to s, then (3.5.1) is also split. On the other hand, since

$$\sigma_R^2(L) = (p_a - g)\mathbf{1}_k \neq 0,$$

then it follows from Theorem 3.2 that the exact sequence (3.5.2) is not split. The proof of the corollary is complete.□

Note that $Q^1 \cong R^1$ always. In fact, it follows from the definition of a Wronski algebra system $F := (F^n, n \geq 0)$ that F^1 is the middle sheaf in the push-out of the infinitesimal $\mathcal{O}_X$-algebra extension

$$0 \longrightarrow \Omega^1 \longrightarrow P^1 \xrightarrow{p^1} \mathcal{O}_X \longrightarrow 0$$

under the canonical homomorphism $\eta^1 : \Omega^1 \to \omega$.

We now give an example where we describe both Q^2 and R^2.

Example 3.6. Let $X \subset \mathbf{P}_k^2$ be the zero scheme of $y^4 - x^3z$. The curve X can be covered by the affine open subsets:

$$U := (z \neq 0) \quad \text{and} \quad V := (x \neq 0).$$

To decribe Q^2 and R^2 we need only describe them as subalgebras of P_K^2 locally on U and V, where K is the field of meromorphic functions of X. Since V is contained in the smooth locus of X, then both Q^2 and R^2 coincide on V with the sheaf of second order principal parts P^2. We will now describe Q^2 and R^2 on U.

We first compute the restriction R_U^2 of R^2 to U. On U we have the identification $U \cong \operatorname{Spec} k[t^3, t^4]$, where $y/z = t^3$ and $x/z = t^4$. It is clear that t is regular on the normalization $\tilde{U} := \operatorname{Spec} k[t]$ of U, and the differential dt is a basis for the sheaf of Kähler differentials on $\tilde{U}$. Moreover, $\omega_U = \mathcal{O}_U(dt)/t^6$ as an $\mathcal{O}_U$-submodule of Ω_K^1. It follows from our description of the sheaf R^2 in Section 2 that R_U^2 is the $\mathcal{O}_U$-subalgebra of P_K^2 generated as a free left $\mathcal{O}_U$-submodule by the basis

$$1, \frac{\delta t}{t^6}, \frac{(\delta t)^2}{t^{12}}. \tag{3.6.1}$$

We now compute the restriction Q_U^2 of Q^2 to U. Assume from now on that char.$k \neq 2$. Let $S := \operatorname{Spec} k[\lambda]$. Let $\mathcal{X} \subset \mathbf{P}_S^2$ be the zero scheme of $y^4 + \lambda xz^3 - x^3z$. Then $\mathcal{X}/S$ is a family of local complete intersection curves, smooth over $S \setminus 0$, and $\mathcal{X}(0) = X$. By [**3**, Prop. 4.3] there is a unique Wronski algebra system $\mathcal{Q} := (\mathcal{Q}^n, n \geq 0)$ on $\mathcal{X}$ over S. Its fibre over 0 is Q. The scheme $\mathcal{X}$ can be covered by the affine open subschemes $\mathcal{U} := (z \neq 0)$ and $\mathcal{V} := (x \neq 0)$. Note that $\mathcal{U}(0) = U$. It is clear that the canonical sheaf $\omega_{\mathcal{X}/S}$ is invertible on $\mathcal{U}$. It follows from the general construction in [**3**, Sections 3 and 4] that there is an isomorphism $\mathcal{Q}_{\mathcal{U}}^2 \cong \mathcal{O}_{\mathcal{U}}[T]/T^3$ as $P_{\mathcal{U}}^2$-algebras, with the structure homomorphism $\psi_{\mathcal{U}}^2\colon P_{\mathcal{U}}^2 \to \mathcal{O}_{\mathcal{U}}[T]/T^3$ being a homomorphism of left $\mathcal{O}_U$-algebras "determined" by the fact that:

$$(y/z + \delta(y/z))^4 + \lambda(x/z + \delta(x/z)) - (x/z + \delta(x/z))^3 = 0. \tag{3.6.2}$$

(In fact, $\psi_{\mathcal{U}}^2$ is determined up to unique automorphism of $\mathcal{O}_{\mathcal{U}}[T]/T^3$.) It "follows" from applying $\psi_{\mathcal{U}}^2$ to (3.6.2) that

$$\begin{aligned} \psi_{\mathcal{U}}^2(\delta(x/z)) =& 4(y/z)^3T + 6(y/z)^2(3(x/z)^2 - \lambda)T^2, \\ \psi_{\mathcal{U}}^2(\delta(y/z)) =& (3(x/z)^2 - \lambda)T + 12(x/z)(y/z)^3T^2. \end{aligned} \tag{3.6.3}$$

Considering the fibre over 0, where we have the identification $(y/z) = t^3$ and $(x/z) = t^4$, the first equation in (3.6.3) becomes

$$\psi_{\mathcal{U}}^2(0)(\delta(t^4)) = 4t^9T + 18t^{14}T^2.$$

Since we also have that $\delta(t^4) = 4t^3\delta t + 6t^2(\delta t)^2$ in P_K^2, then $T = \delta t/t^6 - 3(\delta t)^2/t^7$ in P_K^2. So Q_U^2 is the $\mathcal{O}_U$-subalgebra of P_K^2 generated as a free $\mathcal{O}_U$-submodule by the basis

$$1, \frac{\delta t}{t^6} - 3\frac{(\delta t)^2}{t^7}, \frac{(\delta t)^2}{t^{12}}. \tag{3.6.4}$$

(If char.$k = 2$, then we have to work with a different deformation of X to a family whose general member is smooth. We will not write down the computation of Q_U^2 in this case, but we mention that the same description (3.6.4) works for Q_U^2 when char.$k = 2$ as well.)

Since t^5 is not a regular function on U, we see from (3.6.1) and (3.6.4) that Q^2 and R^2 are isomorphic as P^2-algebras if and only if char.$k = 3$. So the converse to the statement in Corollary 3.5 is true for X. As a matter of fact, I do not know of any curve X where this converse is not true.

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